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PHYSICS AND MATHEMATICS

NYO-7693

ON THE STRUCTURE OF A STEADY HYDROMAGNETIC SHOCK

One-Fluid Theory

Marian H. Rose

October, 1956

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Abstract

The structure of a plane, steady hydromagnetic shock is studied from the point of view of one-fluid continuum theory. Viscosity and heat conductivity are neglected; the electrical conductivity may have any value. The purpose is to derive expressions for the shock profile and thickness in terms of the parameters that enter into the equations of motion of the fluid.

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I. Introduction

The properties of hydromagnetic shocks, considered as sharp discontinuities, are now well known through the work of de Hoffmann and Teller[1], Helfer[2], Lust[3] and Friedrichs[4]. However, due to the relative novelty of the subject, the problem of the internal structure of these shocks has not yet received much attention. Recently, Marshall[5] has treated the case of a plane, steady hydromagnetic shock with constant viscosity and heat conductivity and has obtained profiles for very high and very low electrical conductivities. He shows that among other parameters the thickness is proportional to

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1. De Hoffmann and Teller, Magneto-Hydrodynamic Shocks, Physical Review, 80, 692-703, 1950.

2. Helfer, L., Magneto-Hydrodynamic Shock Waves, Astrophysical Journal, 117, 177-199, 1953.

3. Lust, R., Magneto-Hydrodynamische Stossweller in einem Plasma unendlicher Leitfahigkeit.
Zeitschrift für Naturforschung Band 8a, 277-284, 1953.
Stationare Magneto-Hydrodynamische Stosswellen beliebiger Starke. Zeitschrift für Naturforschung Band 10a, 125, 1955.

4. Friedrichs, K.O., Non-Linear Wave Motion in Magneto-Hydrodynamics, Mimeographed Notes - Los Alamos, 1955.

5. Marshall, W., The Structure of Magneto-Hydrodynamic Shock Waves - Proceedings of the Royal Society, A, 233, 367-376, 1956.

the mean free path. Essentially the same results have been obtained by Sen for the case of infinite conductivity.[6]

In this paper, we shall consider a steady shock in a medium where the state variables may change by significant amounts over the distance of a mean free path, i.e., the mean free path is assumed to be very large. Consequently heat conductivity and viscosity lose their meaning. It is also assumed that the fluid is completely ionized. The simplest case, the one that will be treated here, is that of a plane shock propagating in a direction perpendicular to the applied magnetic field. Obviously, a more satisfactory treatment of hydromagnetic shocks would be based on a two-fluid theory which includes dissipative effects. However, even the simple picture given here enables one to gain some insight into the behavior of these shocks, and interesting properties emerge.

The author wishes to thank Professors Grad, Kolodner and Berkowitz for helpful discussions and criticisms.

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6. Sen, Hari K., Structure of a Magneto-Hydrodynamic Shock Wave in a Plasma of Infinite Conductivity. Physical Review, 102, 5, 1956.

II. Basic Equations

We consider the case of a completely ionized fluid traveling with velocity u with respect to the shock in the positive x direction. The magnetic field is in the z direction. All the state variables, namely the density ρ, the pressure p, the fluid velocity u, the magnetic field B, the current J and the electric field E depend on x alone (Figure 1). Viscosity and heat conductivity are neglected (see Section I); the permeability μ and the electrical conductivity σ are assumed constant. The motion of the fluid is governed by the equations of conservation of mass, momentum and energy together with Maxwell's equations and Ohm's law. To complete this system of equations, it is necessary to add an equation of state for the fluid and to make some assumption about its internal energy. Here we shall assume that the fluid is a perfect gas of monatomic particles. Thus, for a steady state, we obtain the following set of equations

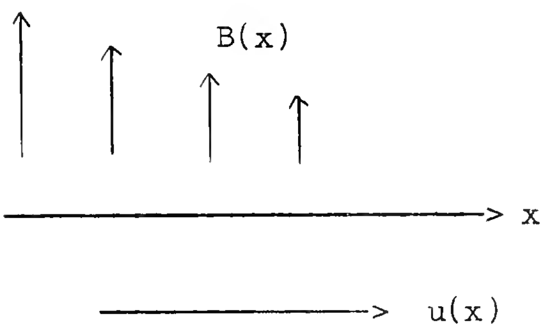

FIG. 1

(1) Conservation of mass $\rho u = M$

(2) Conservation of momentum $\nabla_x(\rho u^2 + p) = J x B$

(3) Conservation of energy $\frac{d}{dx}[u(e + \frac{\rho u^2}{2} + p) = E.J$

where M is a constant of integration and e is the internal energy per unit mass. The symbol ∇_x denotes the x component of the gradient.

Equation of state: $e = \frac{3}{2}RT$ where T is the absolute temperature and R is the gas constant.

Maxwell's equations:

(4) $$\text{curl } E = 0$$

(5) $$\text{curl } \frac{B}{\mu} = J$$

(6) $$\text{div } B = 0$$

(7) Ohm's law: $J = \sigma(E + u \times B)$

using rationalized M.K.S. units. We note that the Hall term in Ohm's law has been dropped since, for this geometry, it is easily shown to vanish identically from the equations into which J enters.[8]

Combining (4), (5) and (7) and then integrating, we find

(8) $$\frac{1}{\mu\sigma}\frac{dB}{dx} = Bu - \triangle$$

where $\triangle$ is an integration constant. Equations (2) and (5) yield, after integration:

(2') $$\rho u^2 + p + \frac{B^2}{2\mu} = \pi$$

π being an integration constant. Lastly, using equation (8) in equation (2) for the conservation of energy leads to

(3') $$u\left(\frac{1}{2}\rho u^2 + \frac{5}{2}p + \frac{B^2}{\mu}\right) = \frac{\mathcal{E}}{2}$$

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8. Grad, H., Notes on Magneto-Hydrodynamics IV, NYO-6486, New York University, 1956.

after integration, $\mathcal{E}$ being another integration constant. We note that in this last equation we have neglected a term of the form $\frac{B}{\mu\sigma}\frac{dB}{dx}$. This approximation is certainly justified for weak shocks or for high conductivities.

Equations (1), (2'), (3') and (8) constitute the basic system of equations. They contain four integration constants which are the parameters of the fluid. Fortunately, this number can be reduced by two by rewriting the equations dimensionlessly.

For this purpose, we introduce the following set of dimensionless variables and parameters:

$$\bar{p} = \frac{p}{\pi}, \quad \bar{B}^2 = \frac{B^2}{2\mu\pi}, \quad \bar{u} = \frac{Mu}{\pi}, \quad \bar{\rho} = \frac{\pi}{M^2}\rho ,$$

$$\bar{\mathcal{E}} = \frac{M\mathcal{E}}{\pi^2}, \quad \bar{\triangle} = \frac{M}{\sqrt{2\mu\pi^3}}\triangle \quad \bar{x} = \frac{\mu\sigma\pi}{M}x, \quad \bar{e} = \frac{M^2 e}{\pi^2}$$

The equations become

(9) $$\bar{\rho}\bar{u} = 1$$

(10) $$\bar{u} + \bar{p} + \bar{B}^2 = 1$$

(11) $$\bar{u}(\bar{u} + 5\bar{p} + 4\bar{B}^2) = \bar{\mathcal{E}}$$

(12) $$\frac{d\bar{B}}{d\bar{x}} = \bar{B}\bar{u} - \bar{\triangle}$$

From here on, remembering that all quantities are dimensionless, we shall omit the bar above the various quantities since there is no longer any possibility of confusion.

There now remain only two parameters, $\mathcal{E}$ and $\triangle$. The parameter $\mathcal{E}$ occurs even when no magnetic field is present and

is closely connected to the shock strength parameter λ of ordinary fluid dynamics. For instance, one possible definition of λ is[9]:

$$\lambda = \sqrt{25 - 16\mathcal{E}}$$

On the other hand, $\triangle$ is an entirely new parameter which depends on the amount of magnetic field present. Since there are two parameters, two auxiliary conditions should be given in order to specify the problem completely. For example, the values of u and B might be assigned in front of the shock. These two conditions, together with the equations, are sufficient to determine the values of the remaining variables and parameters, both in front of and behind the shock. It is now a simple matter to reduce the system of differential equations to a single differential equation for u which may be integrated numerically. However, before having recourse to numerical methods, we shall first describe, in the next two sections, some interesting properties of the shock.

In the last sections, a method for obtaining weak shock profiles will be outlined. One of the most important aspects of the shock, namely its thickness, will also be discussed and its dependence on the various parameters will be shown explicitly.

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9. Grad, H., The Profile of a Steady Shock Wave. Communications on Pure and Applied Mathematics, Vol. V, 257-300, 1952.

III. Discussion of Equations

Since ρ and p are given in terms of u and B by integrated relations, it is possible to express the equations in terms of u and B only. The differential equation (12) for B does not yield any information about the domain of existence of the shock. This information, however, may be obtained from the differential equation for u. From equations (9), (10), (11) and (12), we obtain

$$\left\{\begin{aligned} &(13) \qquad \frac{du}{dx} = \frac{2Bu^2(Bu - \triangle)}{\varepsilon - 4u^2} \\ &(14) \qquad B^2 = 5 - 4u - \frac{\varepsilon}{u} \end{aligned}\right.$$

Of all the possible solutions of this system, the one which represents a shock must satisfy the following conditions: u must be a single-valued function of x and must approach a constant value as x approaches infinity. Letting $\frac{du}{dx} = 0$ entails $Bu = \triangle$. Substituting for B in terms of u in (13) leads to the following cubic for u:

$$u^3 - \frac{5}{4} u^2 + \frac{\varepsilon}{4} u + \frac{\triangle^2}{4} = 0 \tag{15}$$

Since $\triangle^2 > 0$, and since only positive roots of (15) are relevant, the third extraneous root must be negative. The two roots of interest, u_- and u_+, will then represent the values of u at infinity, in front of and behind the shock.

These statements are illustrated in figure 2. The roots, u_+ and u_-, lie at the intersection of $Bu = \Delta$ with the curve $B^2 = 5 - 4u - \frac{\varepsilon}{u}$.

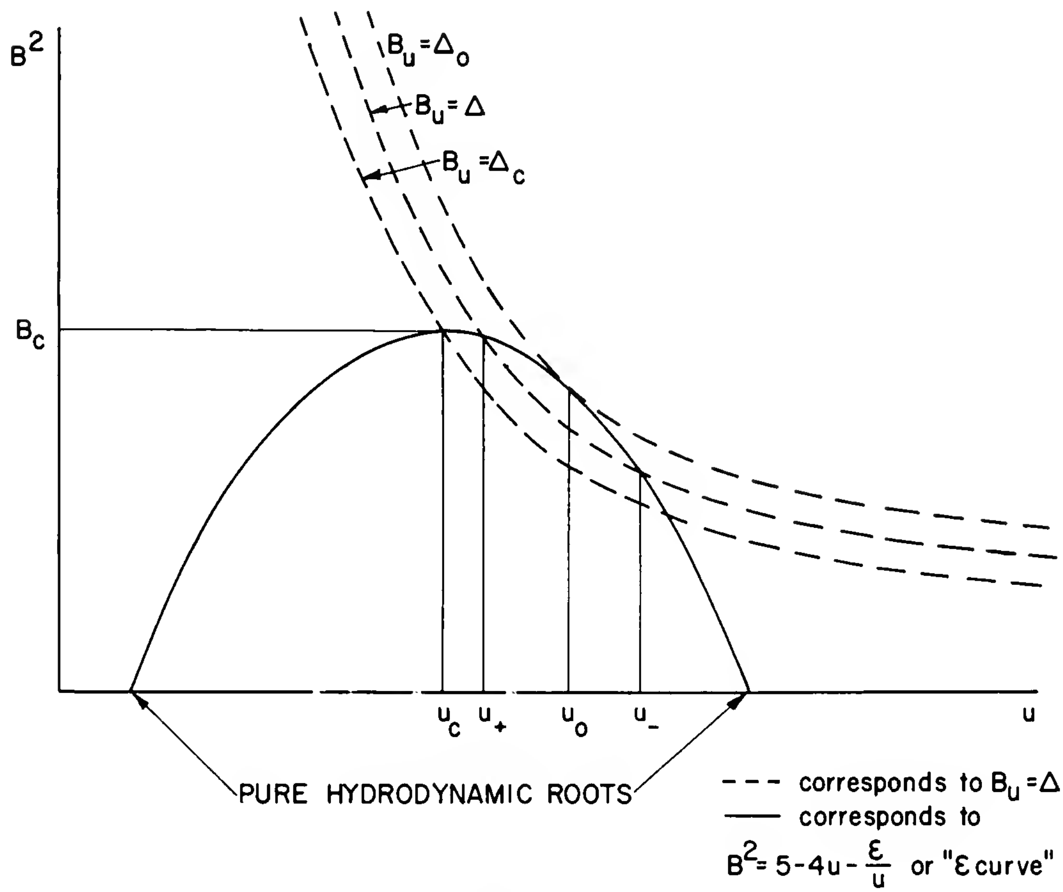


FIG 2

A zero strength shock occurs when the two curves become tangent to each other. Then $u_+ = u_- \equiv u_o(\varepsilon)$. We note that the points of intersection of the "ε curve" with the u axis correspond to the roots for the pure hydrodynamic case and that the roots for the hydromagnetic case always lie between these two.

It is immediately apparent that, for a given ε, $Bu - \triangle_o \leq 0$ where $\triangle_o$ is the value of $\triangle$ for a zero strength shock. Hence, the numerator of $\frac{du}{dx}$ is always negative or zero. On the other hand, the denominator changes sign at $u = \sqrt{\frac{\varepsilon}{4}} \equiv u_c(\varepsilon)$ where $\frac{du}{dx}$ becomes infinite. Consequently, for $u < \sqrt{\frac{\varepsilon}{4}}$, a shock is no longer possible.[10]

10. The value $u = \sqrt{\frac{\varepsilon}{4}}$ also corresponds to the maximum of $B^2 = 5 - 4u - \frac{\varepsilon}{u}$. Hence, no amplification of the magnetic field within the shock is possible. The opposite statement had previously been made (cf. Rose and Grad, Bulletin of the American Physical Society, Series II, Vol. I, No. 1, 1956).

IV. Range of Parameters

It is apparent from the preceding discussion that shocks are obtained only within certain ranges of values of the parameters. Before determining these ranges, we shall introduce a new parameter Λ, related to the shock strength. For reasons of convenience, we shall consider $\mathcal{E}$ and Λ to be the fundamental parameters of which Δ is a known function.

In the pure hydrodynamic case,[11] the values of u, far from the shock, are the roots of a second degree equation, namely,

$$u_{\pm} = \frac{1}{8}\left(5 \pm \sqrt{25 - 16\mathcal{E}}\right) .$$

An obvious choice for the shock strength parameter is then $\lambda = \sqrt{25 - 16\mathcal{E}}$ or some convenient function thereof, since for zero shock strength, $u_+ = u_-$ ($u_o = \frac{5}{8}$ in this case).

In an analogous way, the shock strength parameter for the hydromagnetic case is defined from

$$u_- = u_o + f(\Lambda, \mathcal{E})$$

$$u_+ = u_o - \Lambda$$

where $f(\Lambda, \mathcal{E})$ may be determined when Λ and $\mathcal{E}$ are given.

Some information about the range of the parameters may be obtained from the conditions for a zero strength shock. A necessary condition for such a shock is that $\frac{du}{dx} = 0$ in equation

11. Cf. reference 9.

(13), leading to equation (15) for u. Since, furthermore, we seek two roots of this equation which are equal, positive and real, the third root must also be real. Hence (15) may be expressed in the form

$$(u - u_o)^2 (u + u_o') = 0$$

leading to the following relations between the coefficients:

$$-\frac{5}{4} = u_o' - 2u_o \tag{16}$$

$$\frac{\varepsilon}{4} = u_o^2 - 2u_o u_o' \tag{17}$$

$$\frac{\Delta^2}{4} = u_o' u_o^2 \tag{18}$$

From (16) and (17)

$$u_o = \frac{5/6 \pm \sqrt{25/36 - \varepsilon/3}}{2} \tag{19}$$

From (18), we know that $u_o' > 0$. Substituting for u_o in (18) shows that the "+" sign in the expression for u_o must be taken to obtain a positive value of u_o'. Hence

$$u_o' = -\frac{5}{12} + \sqrt{25/36 - \varepsilon/3}$$

From (18) we obtain

$$\Delta^2 = \left(-\frac{5}{12} + \sqrt{25/36 - \varepsilon/3}\right)\left(5/6 + \sqrt{25/36 - \varepsilon/3}\right)^2 \equiv \Delta_o^2 \tag{20}$$

Both u_o and Δ_o depend on ε alone. Since ε may vary between limits, which will presently be determined, we see that

there exists a one-parameter family of zero strength shocks. As we should expect, the zero strength values for pure hydrodynamical shocks are contained within this family, namely, for $\mathcal{E} = \frac{25}{16}$ we obtain $u_o = \frac{5}{8}$ and $\Delta_o = 0$. The "$\mathcal{E}$ curve" then loses its structure, the points of intersection u_+ and u_- with the u axis coalescing at the point $u = \frac{5}{8}$. In this case, therefore, only a zero strength hydromagnetic shock is possible. From (19) it is obvious that the upper limit of $\mathcal{E}$ is $\frac{25}{16}$. The lower limit is obtained by setting $p = 0$ in the equations of motion far from the shock (p may not be negative). Thus equation (10) becomes $B^2 = 1 - u$ and substituting for B^2 in (14) yields $\mathcal{E} = 4u - 3u^2$. The minimum value of $\mathcal{E}$ may be obtained by differentiating this expression with respect to u. The result is $\mathcal{E} = \frac{4}{3}$. The parameter $\mathcal{E}$ must, therefore, lie within the range

$$\frac{4}{3} \leq \mathcal{E} \leq \frac{25}{16} \quad .$$

On the other hand, Λ may vary from 0 to Λ_{max} where Λ_{max} is the maximum possible shock strength for a given $\mathcal{E}$. This occurs when u_+ is chosen to coincide with u_c, i.e., $\Lambda_{max} = u_o - u_c$. Hence Λ lies within the range

$$0 \leq \Lambda \leq \frac{5/6 + \sqrt{25/36 - \mathcal{E}/3}}{2} - \sqrt{\frac{\mathcal{E}}{4}} \quad .$$

In figure 3, the permissible values of Λ and $\mathcal{E}$ lie within the shaded area. We note that Λ_{max} is a monotonically

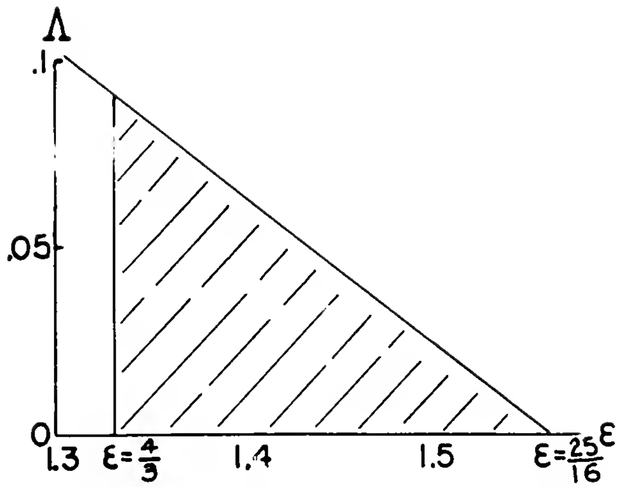


FIG. 3

decreasing function of ε, the maximum value occuring at $\varepsilon = \frac{4}{3}$ for which $\Lambda_{max} = 0.089$. This value of Λ corresponds, therefore, to the strongest possible shock. The range of Δ may be determined in an entirely similar way. It is

$$\Delta_c \leq \Delta \leq \Delta_o$$

where Δ_o is defined by equation (20) and $\Delta_c = u_c B_c = \sqrt{\frac{\varepsilon}{4}}\,(5 - 4\sqrt{\varepsilon})^{1/2}$.

For purposes of future reference, the expressions for $\Delta(\lambda, \varepsilon(u_o))$ and $f(\Lambda, \varepsilon(u_o))$ will be given here. From $\Delta = u_+ B_+$, and, using equations (17) and (18) to express ε in terms of u_o, we obtain

$$(21)\quad \Delta(\Lambda, \varepsilon(u_o)) = (u_o - \Lambda)[5 - 4(u_o - \Lambda) - \frac{4u_o(\frac{5}{2} - 3u_o)}{u_o - \Lambda}]^{1/2} .$$

In order to determine $f(\Lambda, \varepsilon(u_o))$, we allow Δ to vary about its zero strength value in equation (15) while keeping ε fixed ($\varepsilon = \varepsilon_o$). The solutions will also vary about their zero strength value and, therefore, must satisfy

$$[u - (u_o - \Lambda)][u - (u_o + f(\Lambda, \varepsilon))][u + u_o' + \Lambda'] = 0 \quad .$$

Comparison of coefficients leads to

$$(22)\quad f(\Lambda_1, \varepsilon(u_0)) = -\frac{(3u_0 - \Lambda_1 - \frac{\varepsilon}{4})}{2}\left\{1 - \sqrt{1 + \frac{4\Lambda_1}{3u_0 - \Lambda_1 - \frac{\varepsilon}{4}}}\right\} .$$

V. Shock Profile

We shall here describe a method for obtaining weak shock profiles from which values of the thickness may be deduced. The method consists in assuming the following form for $\frac{du}{dx}$; namely,

(23) $$\frac{du}{dx} = \underline{\bar{K}}\,(u,\Lambda,\,\mathcal{E})(u - u_+)(u - u_-)$$

and then expanding $\underline{\bar{K}}\,(u,\Lambda,\mathcal{E})$ in a power series about its value at $\Lambda = 0$. Thus, $K(u,\Lambda,\mathcal{E})$ may be expressed as

(24) $$\underline{\bar{K}}(u,\Lambda,\mathcal{E}) = \underline{\bar{K}}^{(o)}(u,\mathcal{E}) + \Lambda\underline{\bar{K}}^{(1)}(u,\mathcal{E}) + \Lambda^2\underline{\bar{K}}^{(2)}(u,\mathcal{E}) + \ldots$$

The coefficients of Λ are determined by comparison with equation (13). The function $\underline{\bar{K}}(u,\Lambda,\mathcal{E})$ has some physical significance since it can be shown to be related to the shock thickness τ. Although τ may be defined in a variety of ways, for our purposes, the following definition is meaningful:

(25) $$\tau = \frac{u_- - u_+}{\left|\frac{du}{dx}\right|_{max}}$$

where $\left|\frac{du}{dx}\right|_{max}$ is the maximum slope in the profile (cf. figure 4). Assuming $\underline{\bar{K}}$ constant, τ would have the value $\tau = \frac{4}{\underline{\bar{K}}(u_- - u_+)}$. Hence, $\underline{\bar{K}}$ would be inversely proportional to the thickness. In this case, equation (23) integrates to

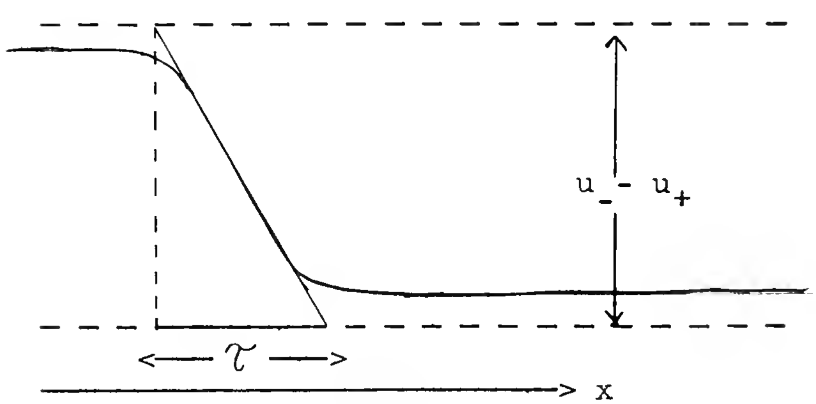


FIG. 4

$$(26) \qquad -\frac{2x}{\tau} = \tanh^{-1} \frac{2u_-(u_- + u_+)}{u_- - u_+}$$

which satisfies the appropriate conditions at $x = \pm \infty$.

Before proceeding with the calculations, it is necessary to make a change of variable, the reason being that the profile degenerates to a straight line for $\Lambda = 0$; u therefore cannot be analytic in Λ in this neighborhood. By introducing the proper variable, w, the profile retains its structure even for zero strength shocks. We define w as follows:

$$(27) \qquad u = u_o + \Lambda w$$

hence, $-1 \leq w \leq \frac{f(\Lambda, \mathcal{E})}{\Lambda}$ where $\frac{f(\Lambda, \mathcal{E})}{\Lambda} \longrightarrow 1$ when $\Lambda \longrightarrow 0$. We also introduce a new coordinate y which may be obtained from x by the transformation $y = \Lambda x$. Equation (23) then becomes

$$(28) \qquad \frac{dw}{dy} = \underline{\bar{K}}(w, \Lambda, \mathcal{E})(w+1)\left(w - \frac{f(\Lambda, \mathcal{E})}{\Lambda}\right) .$$

Similarly, equation (13) transforms into

$$\frac{dw}{dy} = \frac{2(u_o+\Lambda w)^2 \sqrt{5 - \frac{\mathcal{E}}{u_o + \Lambda w} - 4(u_o + \Lambda w)}}{\Lambda^2[\mathcal{E} - 4(u_o + \Lambda w)^2]} \left\{ (u_o + \Lambda w) \cdot \right.$$

(29)

$$\left. \cdot \sqrt{5 - \frac{\mathcal{E}}{u_o + \Lambda w} - 4(u_o + \Lambda w)} - \Delta \right\}$$

$\underline{\bar{K}}^{(o)}$, $\underline{\bar{K}}^{(1)}$, ..., $\underline{\bar{K}}^{(n)}$, ... are then determined by comparison

of equations (28) and (29), the latter first having been expanded in a Taylor series about u_o. Since the main purpose of this paper is to determine the general behavior of the shock thickness rather than to compute the profile to higher order terms, we shall limit ourselves to the first term in the expansion. The result is

$$(30) \qquad \underline{\overline{K}}^{(o)} = \frac{u_o}{2(5-8u_o^2)}\left\{4\left(\frac{5}{2} - u_o\right) + \frac{u_o}{4}\left(7 - \frac{5}{2u_o}\right)^2\right\}$$

a quantity independent of w. In this approximation, w as a function of x is simply

$$(31) \quad w = - \frac{1 + \frac{f(\Lambda, \varepsilon)}{\Lambda}}{2} \tanh\left(\frac{1 + \frac{f(\Lambda, \varepsilon)}{\Lambda}}{2}\right)\underline{\overline{K}}^{(o)} y + \frac{\frac{f(\Lambda, \varepsilon)}{\Lambda} - 1}{2}$$

hence,

$$(32) \quad u = u_o + \Lambda\left\{- \frac{1+ \frac{f(\Lambda, \varepsilon)}{\Lambda}}{2} \tanh \frac{\underline{\overline{K}}^{(o)}\Lambda}{2}\left(\frac{1+ f(\Lambda, \varepsilon)}{\Lambda}\right)x + \frac{\frac{f(\Lambda, \varepsilon)}{\Lambda} - 1}{2}\right.$$

The profile of B is obtained by substituting for u its above value in equation (14). In figures 5, 6, 7 and 8, profiles of u and B are drawn for various values of Λ and ε.

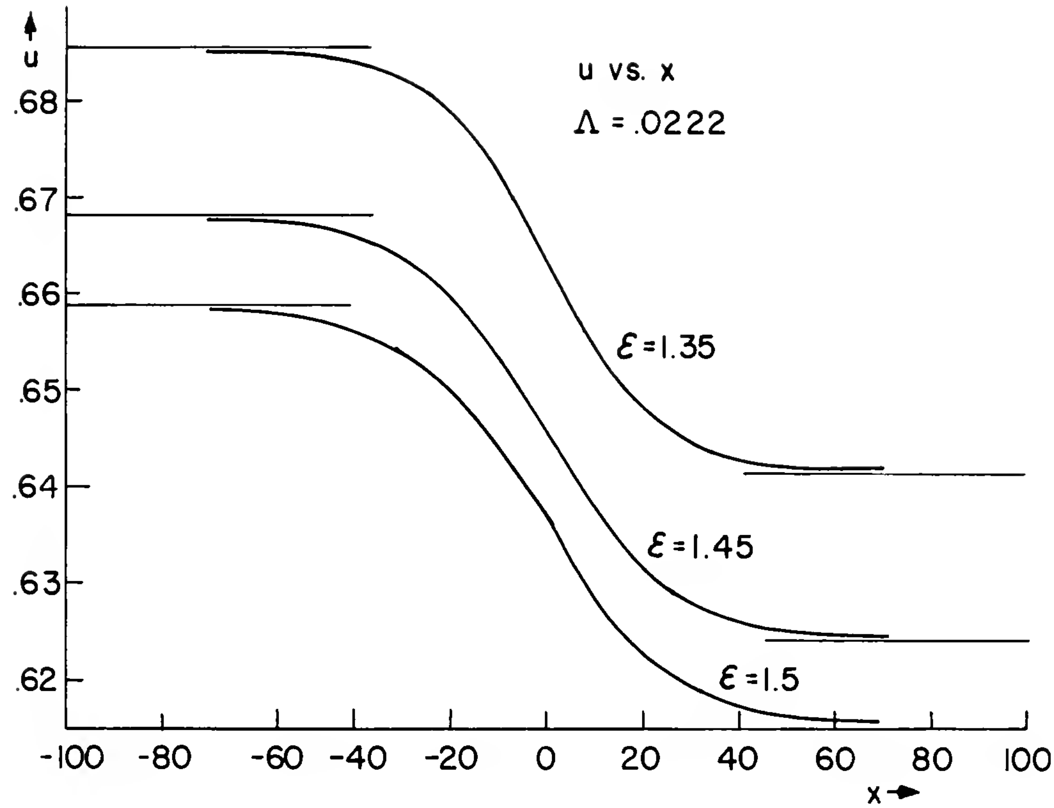

u
u vs. x
Λ = .0222
$\mathcal{E}$ = 1.35
$\mathcal{E}$ = 1.45
$\mathcal{E}$ = 1.5
.68
.67
.66
.65
.64
.63
.62
-100 -80 -60 -40 -20 0 20 40 60 80 100
x

FIG. 5

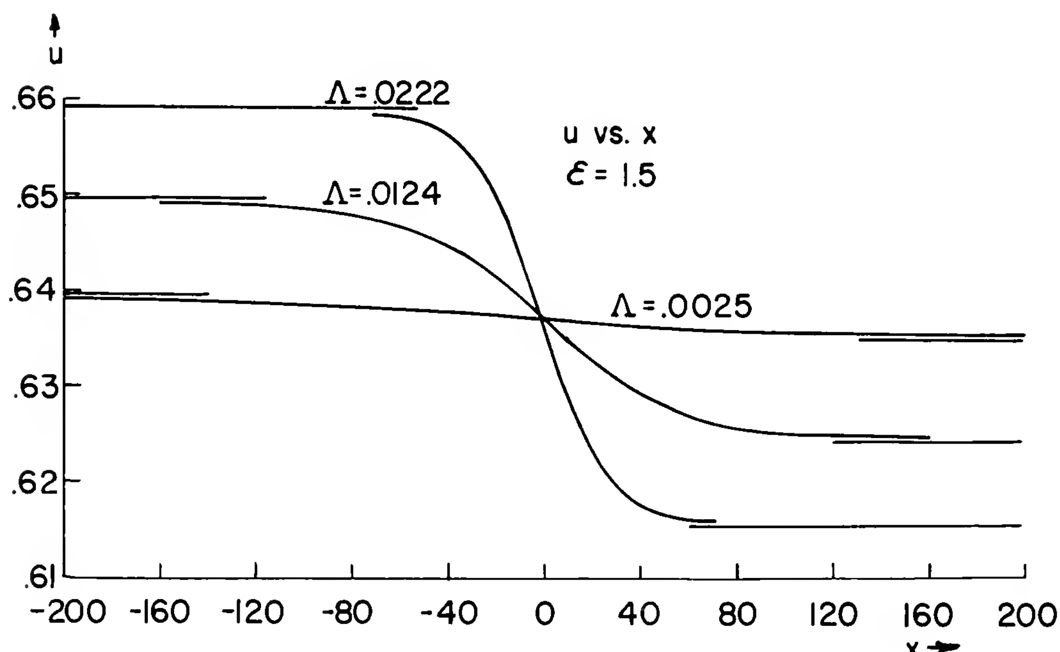
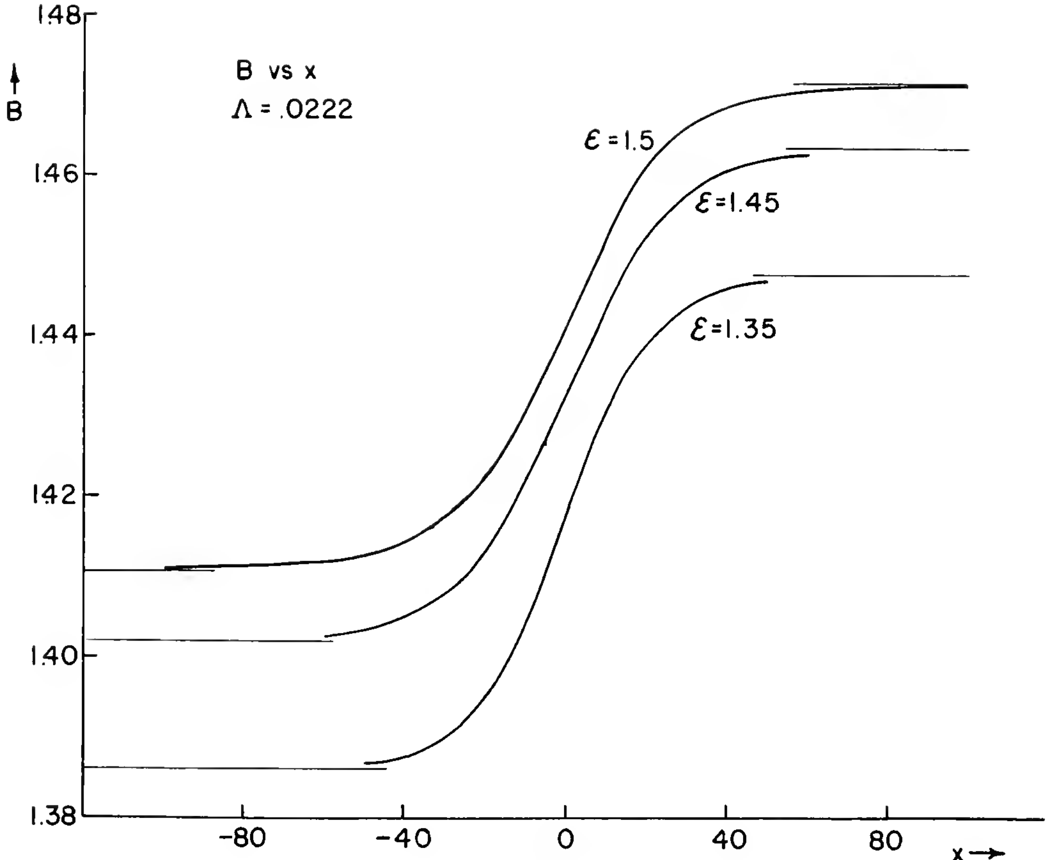
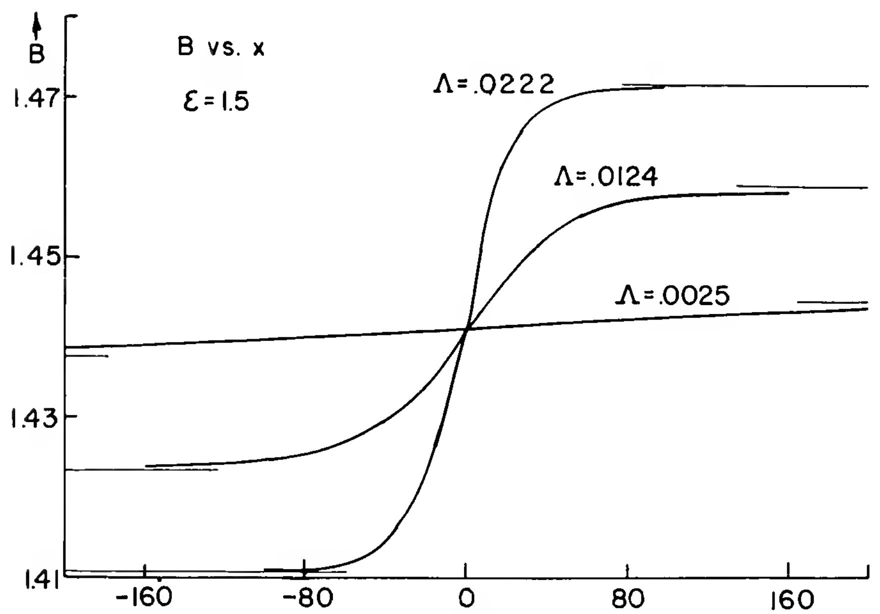

u
Λ = .0222
Λ = .0124
Λ = .0025
u vs. x
$\mathcal{E}$ = 1.5
.66
.65
.64
.63
.62
.61
-200 -160 -120 -80 -40 0 40 80 120 160 200
x

FIG. 6



B vs x
Λ = .0222
B
ε = 1.5
ε = 1.45
ε = 1.35
1.48
1.46
1.44
1.42
1.40
1.38
-80
-40
0
40
80
x

FIG. 7



B vs. x
ε = 1.5
B
Λ = .0222
Λ = .0124
Λ = .0025
1.47
1.45
1.43
1.41
-160
-80
0
80
160
x

FIG. 8

VI. Shock Thickness

Using equation (25), we find that the value of the shock thickness, to a first approximation, is

$$\tau = \frac{4M}{\Lambda\left(\frac{f(\Lambda, \varepsilon)}{\Lambda} + 1\right) \underline{\bar{K}}^{(o)} \mu\sigma\pi} \qquad (33)$$

(The above value for τ has been given the dimensions of a length through multiplication by $\frac{M}{\mu\sigma\pi}$). Replacing M and π by their values from equations (1) and (2') and assuming, as a rough approximation, that $\sigma \propto T^{3/2}$, we conclude that

1) $\tau \propto \frac{1}{B^2}$ keeping p, T and u fixed

2) $\tau \propto T^{-5/2}$ for $p \geq \frac{B^2}{2\mu}$

$\tau \propto T^{-3/2}$ for $p << \frac{B^2}{2\mu}$ keeping p, B and u fixed .

It is worthwhile comparing the shock thickness with the Larmor radius L which might plausibly replace the mean free path as a limiting thickness for the shock when the mean free path becomes very large compared to the dimensions of the apparatus. Now, $L = \frac{mv}{eB}$ where m is the mass of a particle, and v, is its average velocity ($v \propto \sqrt{T}$ where T is the temperature of the gas). Hence,

$$\frac{\tau}{L} = \frac{4}{\underline{\bar{K}}^{(o)}\Lambda\frac{f(\Lambda, \varepsilon)}{\Lambda} + 1} \; \frac{\rho u e B}{\mu\sigma m v\left(Mu + p + \frac{B^2}{2\mu}\right)} .$$

Dividing numerator and denominator by ρ leads to the expression

$\frac{p}{\rho} + \frac{B^2}{2\mu\rho}$ in the denominator which corresponds to the square of one of the three possible speeds at which a hydromagnetic shock may propagate. These have been termed "slow", "intermediate" and "fast" by Friedrichs[12]. For our geometry, the shock can be shown to propagate only at the fast speed, namely, $u_{fast} = \sqrt{\frac{p}{\rho} + \frac{B^2}{2\mu\rho}}$. Assuming that u is approximately equal to u_{fast} we obtain

$$\frac{\tau}{L} \propto \frac{B}{\sigma\sqrt{T\left(\frac{p}{\rho} + \frac{B^2}{2\mu\rho}\right)}} .$$

For small values of $\frac{B^2}{2\mu\rho}$, $\frac{\tau}{L}$ is roughly proportional to B for a given T (the conductivity σ depends on T but is independent of B). For $\frac{B^2}{2\mu\rho} \gg 1$, $\frac{\tau}{L}$ becomes independent of B. In this case, therefore, τ and L are proportional to each other for a given T. In other words the Larmor radius apparently represents a limiting thickness of the shock when the mean free path is infinite. These results are summarized in figure 9 which shows how $\frac{\tau}{L}$ depends on B.

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12. Cf. reference 4.

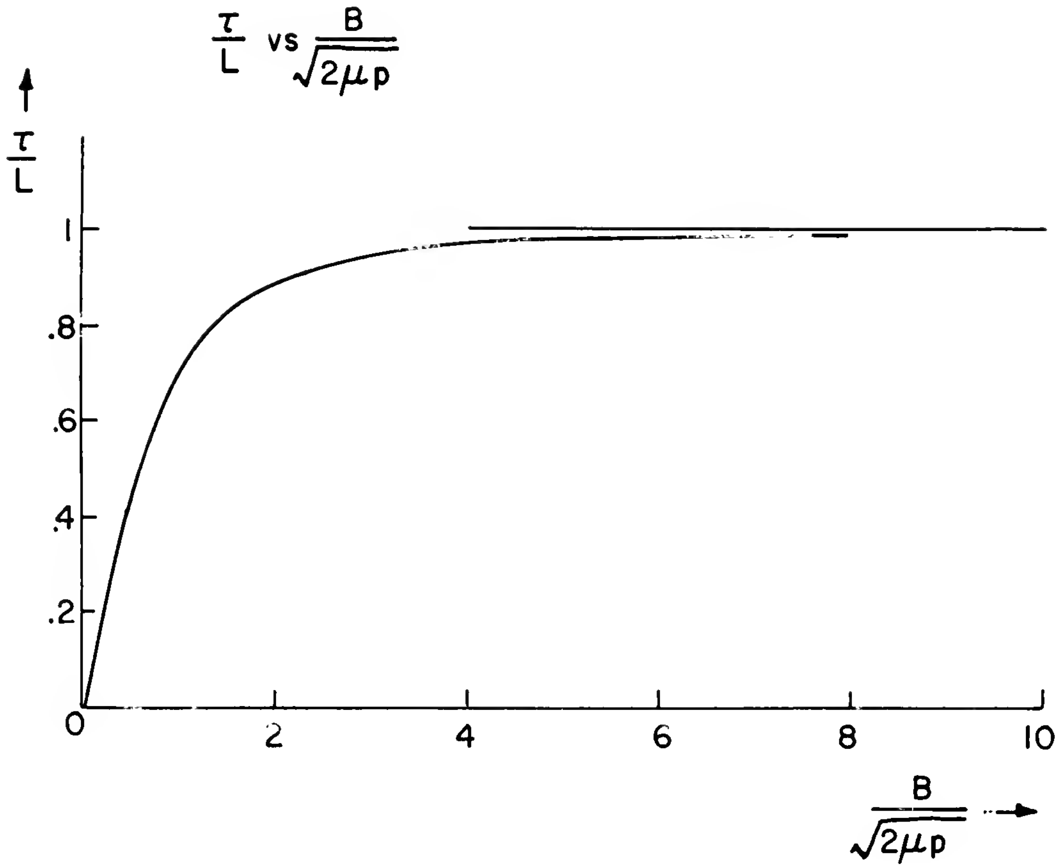
$\frac{\tau}{L}$ vs $\frac{B}{\sqrt{2\mu p}}$
$\frac{\tau}{L}$
1
.8
.6
.4
.2
0
2
4
6
8
10
$\frac{B}{\sqrt{2\mu p}}$

FIG. 9

VII. Comparison with Marshall's Results

In conclusion, it is interesting to compare our results with those of Marshall. His equations include the effects of viscosity and heat conductivity and are therefore applicable to a medium where the gradients are small compared to the mean free path. On the other hand, the case considered here is one where the mean free path is large compared to the dimensions of the region under observation. Hence the usual concepts of viscosity and heat conductivity lose their meaning. In spite of these differences, we shall show that the two theories do indeed meet at a certain point. For this purpose, it will be helpful to recall briefly some of Marshall's results. He shows that the nature of his solution depends on two dimensionless parameters, namely, $\alpha = \frac{3\lambda}{4\gamma R\eta}$ and $\beta = \frac{3\rho c^2}{16\pi\sigma\eta}$ where λ and η are the coefficients of heat conductivity and viscosity respectively, and c, the sound speed of the undisturbed gas. Marshall obtains profiles for $\beta \ll 1$ (σ large) and $\beta \gg \alpha$ (σ small). The first case cannot apply to ours since $\beta \gg 1$ when $\eta \longrightarrow 0$ for any value of σ. On the other hand, the condition $\beta \gg \alpha$ is always met for any σ when both λ and η approach zero. In this case, Marshall shows that two solutions are possible; for one, the fluid speed behind the shock[13] is greater than the local sound speed, for the other, it is less. However,

13. Marshall uses a coordinate system in which the shock is at rest.

it is easily seen that only the direction field of the first solution may coincide with ours. The shock thickness, in this case, is proportional to $\beta \ell = \frac{1}{4\pi\sigma}$ where $\ell = \frac{4\eta}{3\rho c}$ is of the order of magnitude of the mean free path in the undisturbed gas. This is the same result (except for proportionality factors) that was derived in equation (32).

The important difference between the two cases is that Marshall is limited to wide shocks since σ is small, whereas, in our case the shock may become narrow since σ may have any value.

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